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NONLINEAR DYNAMIC ANALYSIS OF ORTHOTROPIC CIRCULAR PLATES

YOGENDRA NATH

Applied Mechanics Department, Indian Institute of Technology, Delhi Hauzkhas, New Delhi-110029, India

and

R. S. ALWAR

Applied Mechanics Department, Indian Institute of Technology, Madras 600036, India

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Abstract—In the present investigation an analytical technique using chebyshev series has been used to study the nonlinear dynamic response of orthotropic circular plates for the both clamped as well as simply supported edge conditions. The influence of orthotropic parameter β^* on the large amplitude response of circular plates, under three types of dynamic loadings namely, step function, sinusoidal and N shaped pulse, has been studied. It is shown that accurate results can be obtained using five terms chebyshev series expansion which is very unlikely in the case of conventional power or trigonometric series.

NOTATIONS

a, h	plate radius and thickness
W, ψ	transverse deflection and stress function
a_{11}, a_{12}	elastic constants
β^*	ratio of elastic constant
q, t	load and time
ν, γ	Poison's ratio and density
$(\sigma_r, \sigma_\theta)$	stress components in radial and tangential directions
$(\epsilon_r, \epsilon_\theta)$	strain components in radial and tangential direction
ρ	nondimensional radius
$\bar{W}, \bar{\psi}$	nondimensional transverse deflection and stress function
P, τ	nondimensional load and time
$T^*(\rho)$	chebyshev polynomials in range $0 \leq \rho \leq 1$
N	number of terms in the series
c_r, b_r; $\bar{W}_r, \bar{\psi}_r$	coefficient of chebyshev series
Subscript J	step of marching variable
Superscript (K)	order of differentiation
Superscript ′	first term of the expansion to be halved
$(\alpha, \beta, \delta, \eta, \zeta)$	Houbolt coefficient
(A, B, C)	Taylor series coefficients.

1. INTRODUCTION

The high diversity and severity of demands as well as of operating conditions particularly in the field of pressure vessels, space and deep water technology imposed on structural elements like plates and shells by today's technology have resulted in the need of nonlinear analysis of these elements made of new materials, such as reinforced plastics and composite materials. Nonlinearity arises due to the large deformation of structures. To treat such cases, the classical linear theory of plates can not be applied adequately and the use of nonlinear theory is quite inevitable. This way it is obvious that this accounts for the wide interest in the engineering and scientific community arisen during the last two decades in substantiating, developing and generalising theoretical methods for the rational analysis of plates and shells. Nonlinear static analysis of rectangular and elliptical orthotropic plates has been carried out by Chia *et al.*[1, 2] and Basu *et al.*[3]. Static post buckling behaviour of orthotropic annular plates has been studied by Uthegennant[4] and Huang[5].

As a consequence of the impetuous development of modern aircraft and space technology, more attention is being paid to the study of dynamic behaviour of elastic structures in conditions of mutual interaction of elastic and inertia forces than the static analysis of structures. The amount of literature available pertaining to the large amplitude response of orthotropic structures is quite limited. Nonlinear dynamic analysis of orthotropic shells has been carried out by Stephens[6] using finite difference and Nowinski[7, 8] using Galerkin techniques. Nonlinear response of rectangular plates is studied by Alwar *et al.*[9] using rate form linearization technique. Large amplitude vibrations of orthotropic circular plate have been studied by Huang[10, 11] using Ritz-Kantorovich method.

The purpose of this investigation is to present an analytical approach to the study of nonlinear static and dynamic response of orthotropic circular plates. The governing nonlinear differential equations are expressed in terms of stress function and transverse deflection. These equations have been integrated space-wise using chebyshev polynomials[12, 13] and time-wise using implicit Houbolt scheme[14] and the influence of orthotropic parameter β^* on the large static and dynamic amplitude response has been investigated for both clamped and simply supported circular plates under various types of dynamic loads. To the author's knowledge, there is no available results for the large amplitude response of orthotropic circular plates under transient loads.

2. MATHEMATICAL FORMULATION

Considering the cylindrically orthotropic material whose axis of orthotropy coincides with the Z axis of the coordinate system which is the axis of symmetry of the circular plate, the governing differential eqns[5] can be expressed as

$$\left(\nabla^2 - \frac{*}{r^2}\right)\psi + \frac{h}{ra_{11}}\left(\frac{\partial W}{\partial r}\right)^2 = 0 \tag{1}$$

$$D\left(\frac{\partial^4 W}{\partial r^4} + \frac{2}{r}\frac{\partial^3 W}{\partial r^3} - \frac{\beta^*}{r^2}\frac{\partial^2 W}{\partial r^2} + \frac{\beta^*}{r^3}\frac{\partial W}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\psi\frac{\partial W}{\partial r}\right) = q(r,t) - \gamma h\frac{\partial^2 W}{\partial t^2} \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \quad \nabla^2 = \nabla^2 \cdot \nabla^2$$

$$\beta^* = a_{22}/a_{11}$$

ψ = stress function, which has got the relationship with the inplane forces N_r and N_θ as

$$N_r = \frac{\psi}{r}$$

$$N_\theta = \frac{\partial\psi}{\partial r}$$

$$D = \frac{h^3}{12a_{11}(\beta^* - \nu^2)} \tag{3a}$$

$$\nu = -a_{12}/a_{11}.$$

The constitutive relationship for cylindrically orthotropic plate follows the following relations

$$\epsilon_\theta = a_{11}\sigma_\theta + a_{12}\sigma_r$$

$$\epsilon_r = a_{12}\sigma_\theta + a_{22}\sigma_r. \tag{3b}$$

By introducing the following nondimensional quantities

$$\rho = r/a$$

$$\bar{W} = W/a$$

$$\bar{\psi} = \frac{a_{11}}{h.a}(\beta^* - \nu^2)\cdot\psi \tag{4}$$

$$P = qa^3/D$$

$$\tau = \sqrt{\frac{D}{\gamma h.a^4}} \cdot t$$

eqns (1) and (2) can be written as

$$\rho^2\left(\nabla^2 - \frac{\beta^*}{\rho^2}\right)\bar{\psi} + \left(\frac{\beta^* - \nu^2}{2}\right)\rho \cdot \left(\frac{\partial \bar{W}}{\partial \rho}\right)^2 = 0 \tag{5}$$

$$\left(\rho^3 \frac{\partial^4 \bar{W}}{\partial \rho^4} + 2\rho^2 \frac{\partial^3 \bar{W}}{\partial \rho^3} - \rho\beta^* \frac{\partial^2 \bar{W}}{\partial \rho^2} + \beta^* \frac{\partial \bar{W}}{\partial \rho}\right) - 12(a/h)^2\rho^2 \frac{\partial}{\partial \rho}\left(\bar{\psi}\frac{\partial \bar{W}}{\partial \rho}\right) = \rho^3\left(P - \frac{\partial^2 \bar{W}}{\partial \tau^2}\right). \tag{6}$$

The boundary conditions in dimensionless form become

(a) Clamped edge

$$\rho = 1 \qquad \bar{W} = 0, \qquad \frac{\partial \bar{W}}{\partial \rho} = 0, \qquad \rho\frac{\partial \bar{\psi}}{\partial \rho} - \nu\bar{\psi} = 0 \tag{7}$$

(b) Simply supported edge

$$\rho = 1 \qquad \bar{W} = 0, \qquad \rho\frac{\partial^2 \bar{W}}{\partial \rho^2} + \nu\frac{\partial \bar{W}}{\partial \rho} = 0, \qquad \rho\frac{\partial \bar{\psi}}{\partial \rho} - \nu\bar{\psi} = 0 \tag{8}$$

(c) Symmetry conditions at the centre

$$\rho = 0 \qquad \frac{\partial \bar{W}}{\partial \rho} = 0, \qquad \frac{\partial^3 \bar{W}}{\partial \rho^3} = 0, \qquad \bar{\psi} = 0. \tag{9}$$

3. ANALYSIS

The nonlinear eqns (5) and (6) are linearized using Taylor series[13]. Expressing one of the product terms, constituting the nonlinearity, in Taylor series expansion and using the backward difference scheme, eqns (5) and (6) can be expressed at step J

$$\rho^2\left(\nabla^2 - \frac{\beta^*}{\rho^2}\right)\bar{\psi}_J + \frac{\beta^* - \nu^2}{2}\rho\left(\frac{\partial \bar{W}}{\partial \rho}\right)_J\left\{A\left(\frac{\partial \bar{W}}{\partial \rho}\right)_{J-1} + B\left(\frac{\partial \bar{W}}{\partial \rho}\right)_{J-2} + C\left(\frac{\partial \bar{W}}{\partial \rho}\right)_{J-3}\right\} = 0 \tag{10}$$

$$\left\{\rho^3\frac{\partial^4\bar W}{\partial\rho^4}+2\rho^2\frac{3\bar W}{3}-\rho\beta^*\frac{\partial^2\bar W}{\partial\rho^2}+\beta^*\frac{\partial\bar W}{\partial\rho}\right\}_J$$

$$-12(a/h)^2\rho^2\left[\left(\frac{\partial\bar\psi}{\partial\rho}\right)_J\left\{A\left(\frac{\partial\bar W}{\partial\rho}\right)_{J-1}+B\left(\frac{\partial\bar W}{\partial\rho}\right)_{J-2}+C\left(\frac{\partial\bar W}{\partial\rho}\right)_{J-3}\right\}\right.$$

$$\left.+\bar\psi_J\left\{A\left(\frac{\partial^2\bar W}{\partial\rho^2}\right)_{J-1}+B\left(\frac{\partial^2\bar W}{\partial\rho^2}\right)_{J-2}+C\left(\frac{\partial^2\bar W}{\partial\rho^2}\right)_{J-3}\right\}\right]$$

$$=\rho^3\left(P-\frac{\partial^2\bar W}{\partial\tau^2}\right). \tag{11}$$

Where (A, B, C) are the coefficients of Taylor series linearization technique and which take the following value during the initial increment of time

$$\begin{aligned}
&J=1, \\
&\qquad A=B=C=0\\
&J=2,\\
&\qquad A=2.0\\
&\qquad B=C=0\\
&J=3\\
&\qquad A=-2\\
&\qquad B=2.5\\
&\qquad C=0.0\\
&J>3\\
&\qquad A=2.5\\
&\qquad B=-2.0\\
&\qquad C=0.5.
\end{aligned}\tag{12}$$

The integration of eqns (10) and (11) is carried out space-wise using chebyshev polynomials and time wise using Houbolt scheme. A method of solution employing Chebyshev's Theory for the problem of Nonlinear Analysis of Isotropic Circular Plates subjected to Static and dynamic loads and the corresponding properties for Chebyshev polynomials have been discussed earlier by Alwar and Yogendra Nath[12, 13].

The deflection function $\bar W(\rho,\tau)$, stress function $\bar\psi(\rho,\tau)$ and load $P(\rho,\tau)$ can be expressed in Chebyshev polynomials as

$$\begin{aligned}
\bar W &= \sum_{r=0}^{N}{}' \bar W_r T_r^*(\rho)\\
\bar\psi &= \sum_{r=0}^{N}{}' \bar\psi_r T_r^*(\rho)\\
P &= \sum_{r=0}^{N}{}' P_r T_r^*(\rho).
\end{aligned}\tag{13}$$

where, $\bar W_r$, $\bar\psi_r$ and P_r are the coefficients of chebyshev polynomials. $T_r^*(\rho)$ is the chebyshev series in the range $0\leq\rho\leq1$ and supercript$'$ denotes the first term of the series to be halved.

Substituting eqn (3) into eqns (10) and (11), it is obtained at any step J.

$$\sum_{r=0}^{N-2}{}' \left[\left\{ \frac{1}{16}\bar{\psi}^{(2)}_{r+2} + \frac{1}{4}\bar{\psi}^{(2)}_{r+1} + \frac{3}{8}\bar{\psi}^{(2)}_{r} + \frac{1}{4}\bar{\psi}^{(2)}_{r-1} + \frac{1}{16}\bar{\psi}^{(2)}_{r-2} \right\}_J + \left\{ \frac{1}{4}\bar{\psi}^{(1)}_{r+1} + \frac{1}{2}\bar{\psi}^{(1)}_{r} + \frac{1}{4}\bar{\psi}^{(1)}_{r-1} \right\}_J - \beta^* \left\{ \bar{\psi}_r \right\}_J \right.$$
$$\left. + \left(\frac{\beta^* - \nu^2}{2} \right) \left\{ \frac{1}{4}\bar{c}_{r+1} + \frac{1}{2}\bar{c}_r + \frac{1}{4}\bar{c}_{r-1} \right\}_J \right] T^*_r = 0 \qquad (14)$$

$$\sum_{r=0}^{N-4}{}' \left| \left[\left\{ \frac{1}{64}\bar{W}^{(4)}_{r+3} + \frac{3}{32}\bar{W}^{(4)}_{r+2} + \frac{15}{64}\bar{W}^{(4)}_{r+1} + \frac{5}{16}\bar{W}^{(4)}_{r} + \frac{15}{64}\bar{W}^{(4)}_{r-1} + \frac{3}{32}\bar{W}^{(4)}_{r-2} + \frac{1}{64}\bar{W}^{(4)}_{r-3} \right\}_J \right.\right.$$
$$+ 2\left\{ \frac{1}{16}\bar{W}^{(3)}_{r+2} + \frac{1}{4}\bar{W}^{(3)}_{r+1} + \frac{3}{8}\bar{W}^{(3)}_{r} + \frac{1}{4}\bar{W}^{(3)}_{r-1} + \frac{1}{16}\bar{W}^{(3)}_{r-2} \right\}_J$$
$$- \beta^* \left\{ \frac{1}{4}\bar{W}^{(2)}_{r+1} + \frac{1}{2}\bar{W}^{(2)}_{r} + \frac{1}{4}\bar{W}^{(2)}_{r-1} \right\}_J + \beta^* \left\{ \bar{W}^{(1)}_r \right\}_J$$
$$\left. - 12\left(\frac{a}{h}\right)^2 \left\{ \frac{1}{16}\bar{b}_{r+2} + \frac{1}{4}\bar{b}_{r+1} + \frac{3}{8}\bar{b}_r + \frac{1}{4}\bar{b}_{r-1} + \frac{1}{16}\bar{b}_{r-2} \right\}_J \right] T^*_r$$
$$- \left[\left\{ \frac{1}{64}T^*_{r+3} + \frac{3}{32}T^*_{r+2} + \frac{15}{64}T^*_{r+1} + \frac{5}{16}T^*_r \right.\right.$$
$$\left. + \frac{15}{64}T^*_{r-1} + \frac{3}{32}T^*_{r-2} + \frac{1}{64}T^*_{r-3} \right\}$$
$$\left.\left. \times \left\{ P_J - \frac{1}{\Delta\tau^2}\left(\alpha\bar{W}_{r,J} + \beta\bar{W}_{r,J-1} + \delta\bar{W}_{r,J-2} + \eta\bar{W}_{r,J-3} + \gamma \right) \right\} \right] \right| = 0. \qquad (15)$$

Note that inertia term is expressed in the Houbolt scheme in the above eqn (15).

Where,

$$c_{r,J} = \frac{1}{2}\sum_{i=0}^{N-1}{}' \left[\left\{ \bar{W}^{(1)}_{r+i} + \bar{W}^{(1)}_{r-i} \right\}_L \left\{ A\bar{W}^{(1)}_{r,J-1} + B\bar{W}^{(1)}_{r,J-2} + C\bar{W}^{(1)}_{r,J-3} \right\} \right] \qquad (16)$$

$$\bar{b}_{r,J} = \frac{1}{2}\sum_{i=0}^{N-1}{}' \left[\left\{ \bar{\psi}_{r+i} + \bar{\psi}_{r-i} \right\}_J \left\{ A\bar{W}^{(2)}_{r,J-1} + B\bar{W}^{(2)}_{r,J-2} + C\bar{W}^{(2)}_{r,J-3} \right\} \right.$$
$$\left. + \left\{ \bar{\psi}^{(1)}_{r+i} + \bar{\psi}^{(1)}_{r-i} \right\}_J \left\{ A\bar{W}^{(1)}_{r,J-1} + B\bar{W}^{(1)}_{r,J-2} + C\bar{W}^{(1)}_{r,J-3} \right\} \right] \qquad (17)$$

($\alpha, \beta, \delta, \eta$ and ζ) are the coefficients of Houbolt scheme for the evaluation of inertia term. The evaluation procedure is given the Appendix A.

Similarly the boundary and symmetry conditions take the following form

(a) $\rho = 1$

$$\sum_{r=0}^{N}{}' \bar{W}_r T^*_r = 0$$
$$\sum_{r=0}^{N-1}{}' \bar{W}^{(1)}_r T^*_r = 0 \qquad (18)$$
$$\sum_{r=0}^{N-1}{}' \left\{ \frac{1}{4}\bar{\psi}^{(1)}_{r+1} + \frac{1}{2}\bar{\psi}^{(1)}_r + \frac{1}{4}\bar{\psi}^{(1)}_{r-1} \right\} T^*_r - \nu \sum_{r=0}^{N}{}' \bar{\psi}_r T^*_r = 0$$

(b) $\rho = 1$

$$\sum_{r=0}^{N}{}' \bar{W}_r T^*_r = 0$$

$$\sum_{r=0}^{N-2}{}' \left\{\frac{1}{4}\bar{W}^{(2)}_{r+1}+\frac{1}{2}\bar{W}^{(2)}_{r}+\frac{1}{4}\bar{W}^{(2)}_{r-1}\right\}T^*_r+\nu\sum_{r=0}^{N-1}\bar{W}^{(1)}_rT^*_r=0$$
$$\sum_{r=0}^{N-1}{}' \left\{\frac{1}{4}\bar{\psi}^{(1)}_{r+1}+\frac{1}{2}\bar{\psi}^{(1)}_{r}+\frac{1}{4}\bar{\psi}^{(1)}_{r-1}\right\}T^*_r-\nu\sum_{r=0}^{N}{}'\bar{\psi}_rT^*_r=0 \tag{19}$$

(c) $\rho = 0$

$$\sum_{r=0}^{N}{}' \bar{W}^{(1)}_r \quad T^*_r = 0$$
$$\sum_{r=0}^{N}{}' \bar{W}^{(3)}_r \quad T^*_r = 0 \tag{20}$$
$$\sum_{r=0}^{N}{}' \bar{\psi}_r \quad T_r = 0.$$

The derivative coefficients of chebyshev polynomials T^*_r have the following recurrence relation

$$a^{(k+1)}_{r-1}-a^{(k+1)}_{r+1}=4\,r\,a^{(k)}_r \tag{21}$$

where superscripts $(k+1)$, (k) denote the order of derivative.

Equations (14) and (15) are the generating equations for the evaluation of the unknown coefficients $\bar{W}_r$ and $\bar{\psi}_r$ of chebyshev series T^*_r.

Using eqn (21), these equations can be expressed in terms of $\bar{W}_r$, $\bar{W}_{r+1}\ldots$, $\bar{\psi}_r$, $\bar{\psi}_{r+1}\ldots$, etc. Now by equating the coefficients of T^*_r for $r=0, 1,\ldots N$, a set of simultaneous algebraic equations in terms of $\bar{W}_0$, $\bar{W}_1,\ldots \bar{W}_N$, $\bar{\psi}_0$, $\bar{\psi}_1,\ldots \bar{\psi}_N$ is obtained and which can be expressed in matrix form

$$[\bar{\delta}_{ij}]\{\bar{\zeta}\}=\{\bar{\beta}\} \tag{22}$$

where,

$[\bar{\delta}_{ij}]$ = coefficient matrix

$\{\bar{\zeta}\}$ = Unknown vector,

$\{\bar{\beta}\}$ = Load vector with inertia term.

Equation (22) is solved without iterations.

4. RESULT AND DISCUSSIONS

Nonlinear static analysis has been carried out for the clamped as well as simply supported orthotropic circular plates under uniformly distributed load and the influence of β^* on the deflection has been studied. Three values of β^*, namely $\beta^* = 1.0$, 0.75 and 0.50 are considered in the present investigation. $\beta^* = 1.0$ denotes the isotropic case. Figure 1 shows the results for clamped edge condition. It can be seen that the results for $\beta^* = 1.0$, isotropic case agrees well with the results given by Way[15]. This can also be taken as the check on the present analysis. Figure 2 shows the results for the simply supported edge condition. It can be noted that higher the value of β^* higher is the value of deflection.

Dynamic analysis has been carried out for three types of dynamic loadings, namely, step function, sinusoidal pulse and 'N' shaped pulse and the influence of β^* on the large amplitude response has been investigated. Figure 3 shows the results for the clamped edge condition under a step function load of intensity $qa^4/Eh^4 = 10$ for $\beta^* = 1.0$, 0.75 and 0.50. Similar results for the simply supported edge condition are plotted in Fig. 4. The results for influence of β^* on the large amplitude response of a clamped edge plate subjected to sinusoidal and symmetric 'N' shaped loadings are shown in Figs. 5 and 6 respectively. It can be noticed from Figs. 3–6 that

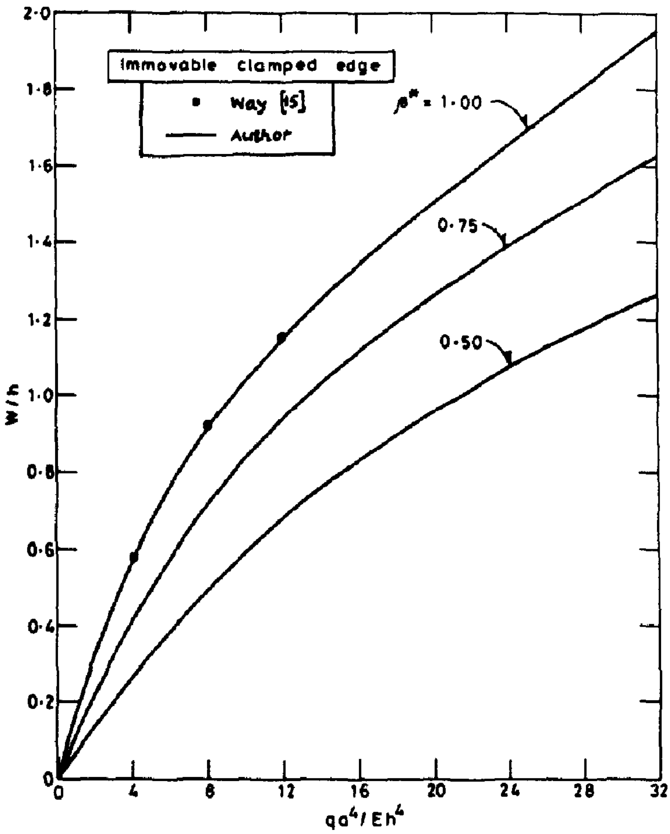

Fig. 1. Influence of β^* on the nonlinear static deflection of circular plate.

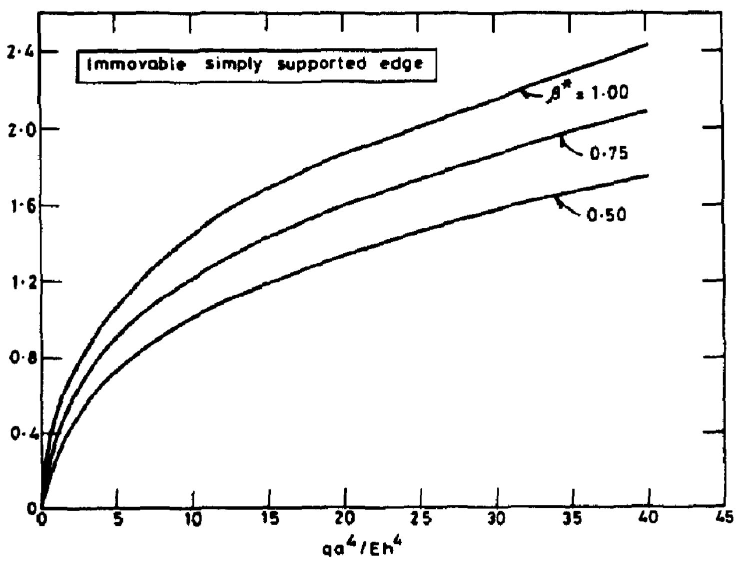

Fig. 2. Influence of β^* on the nonlinear static deflection of circular plate.

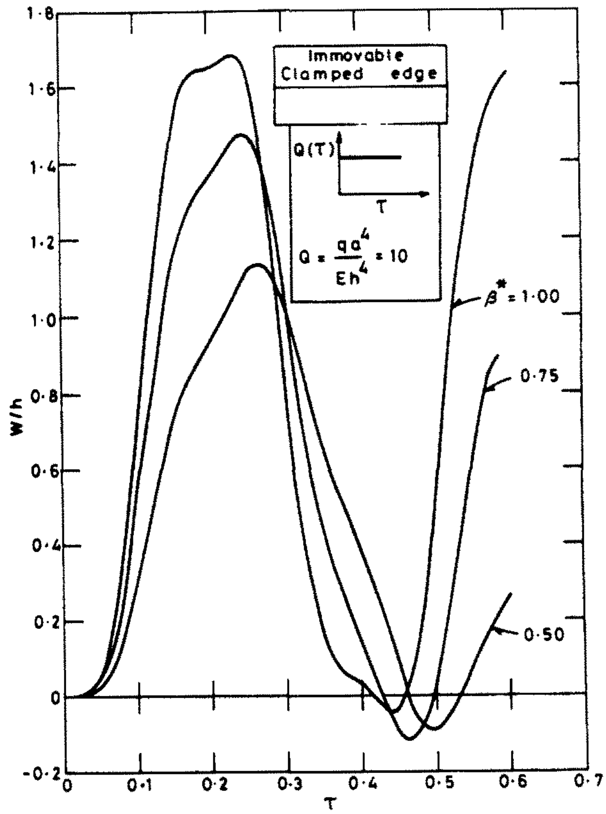

Fig. 3. Influence of β^* on the dynamic response of circular plate to step function load.

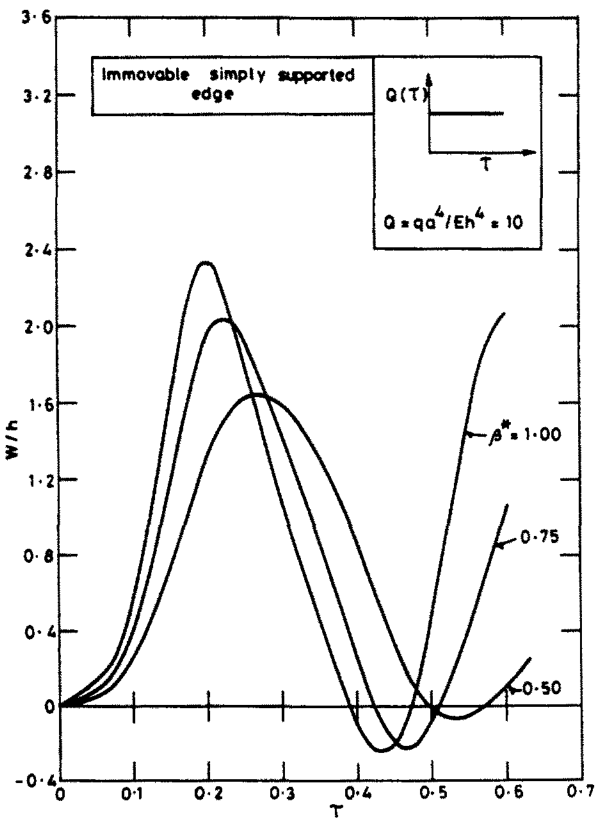

Fig. 4. Influence of β^* on the dynamic response of circular plate to step function load.

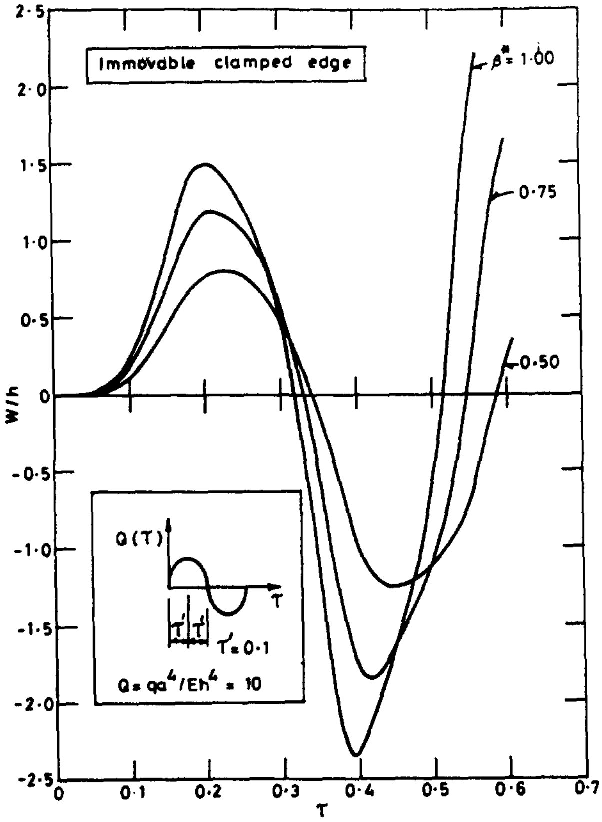

Fig. 5. Influence of β^* on the dynamic response of circular plate to sine wave pulse.

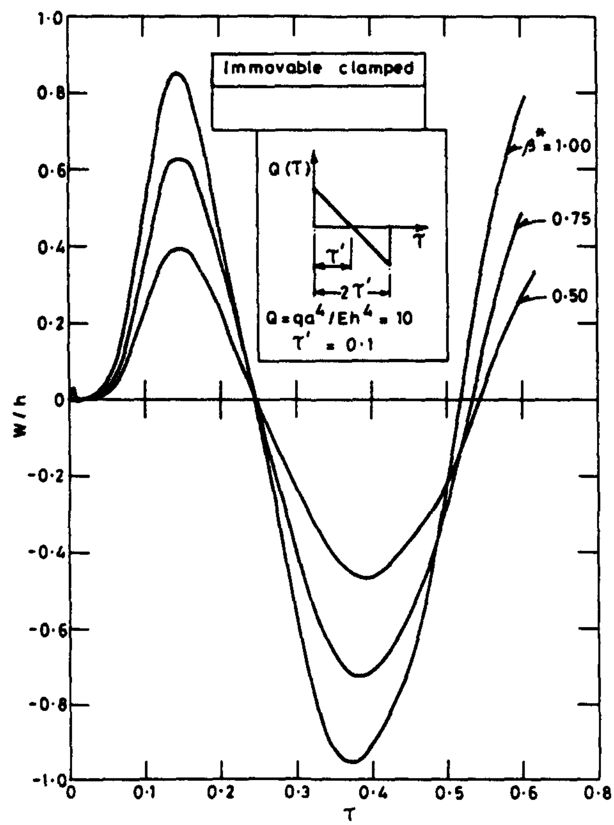

Fig. 6. Influence of β^* on the dynamic response of circular plate to 'N' wave pulse.

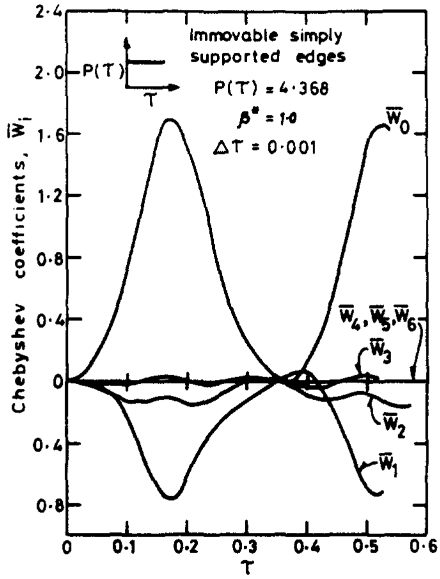

Fig. 7. Convergence of chebyshev series.

lesser the value of β^* lesser is the dynamic response and maximum response is observed for $\beta^* = 1.0$, which corresponds to the isotropic case.

In the present analysis only seven term expansion of chebyshev series for both the function $\bar{W}$ and $\bar{\psi}$, have been considered. The coefficients $\bar{W}_0$, $\bar{W}_1$, $\bar{W}_2 \ldots \bar{W}_6$ are plotted in Fig. 7. It can be seen that the higher order coefficients $\bar{W}_5$ and $\bar{W}_6$ are negligible and therefore a five term expansion would lead to accurate results, which is very unlikely in the case of conventional power or trigonometric series.

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APPENDIX A

The following relations as suggested by Houbolt are used to start the recurrence process

$$\left(\frac{\partial^2 \bar{W}}{\partial \tau^2}\right)_{r,J} = \frac{1}{\Delta \tau^2}\{\bar{W}_{r,J+1} - 2\bar{W}_{r,J} + \bar{W}_{r,J-1}\} \tag{A1}$$

$$\left(\frac{\partial \bar{W}}{\partial \tau}\right)_{r,J} = \frac{1}{6\Delta \tau}\{2\bar{W}_{r,J+1} + 3\bar{W}_{r,J} - 6\bar{W}_{r-1,J} + \bar{W}_{r,J-2}\}. \tag{A2}$$

The initial condition for step function loading are as follows:

$$J = 0$$

$$\bar{W}_{r,0} = 0$$

$$\left(\frac{\partial \bar{W}}{\partial \tau}\right)_{r,0} = 0. \tag{A3}$$

Substituting eqn (A3) in eqn (4) it is obtained

$$\left(\frac{\partial^2 \bar{W}}{\partial \tau^2}\right)_{r,0} = P(\tau). \tag{A4}$$

Substituting eqns (A3) and (A4) in eqn (A1) and (A2) the fictitious coefficient at negative time step are obtained as

$$\bar{W}_{r,-1} = P(\tau) \quad \Delta\tau^2 - \bar{W}_{r,1} \tag{A5}$$

$$\bar{W}_{r,-2} = 6P(\tau) \quad \Delta\tau^2 - 8\bar{W}_{r,1}. \tag{A6}$$

The values of fictitious coefficients $\bar{W}_{r,-1}$ and $\bar{W}_{r,-2}$ as expressed in eqns (A5) and (A6) are substituted in the following equations to yield the coefficients (α, β, δ, η and ζ) during the initial and for all increment of time, i.e.

$$\left(\frac{\partial^2 \bar{W}}{\partial \tau^2}\right)_{r,J} = \frac{1}{\Delta\tau^2}\{\alpha \bar{W}_{r,J} + \beta \bar{W}_{r,J-1} + \delta \bar{W}_{r,J-2} + \eta \bar{W}_{r,J-3} + \zeta\} \tag{A7}$$

$J = 1$

$$\alpha = \frac{6}{\Delta\tau^2}, \beta = \delta = \eta = 0; \zeta = -2P(\tau)$$

$J = 2$

$$\alpha = \frac{2}{\Delta\tau^2}, \beta = -\frac{4}{\Delta\tau^2}, \delta = \eta = 0, \zeta = -P(\tau)$$

$J = 3$

$$\alpha = \frac{2}{\Delta\tau^2}, \beta = -\frac{5}{\Delta\tau^2}, \delta = \frac{4}{\Delta\tau^2}, \eta = \zeta = 0$$

$J > 3$

$$\alpha = \frac{2}{\Delta\tau^2}, \beta = -\frac{5}{\Delta\tau^2}, \delta = \frac{4}{\Delta\tau^2}, \eta = -\frac{1}{\Delta\tau^2}, \zeta = 0.$$